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## The pre-hull number and lexicographic product

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## ABSTRACT

Recently the invariant (copoint) pre-hull number  $ph(G)$  of a graph  $G$  that measures the nonconvexity of a convex space was introduced in [15]. We introduce a similar invariant called the convex pre-hull number which is a natural upper bound for the copoint pre-hull number and consider in this work both on the lexicographic product of graphs. We present exact values with respect to properties of the factors.

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## 1. Introduction and preliminaries

Polat and Sabidussi have recently introduced in [15] a numerical measure of nonconvexity of convex spaces—a (copoint) pre-hull number. They had a restriction and have observed only copoints  $C \subset V(G)$  for a graph  $G$ , i.e. maximal convex sets with the property that they do not include some vertex of  $G$ . On the other hand the most general approach to the topic is due to Harary and Nieminen [10] and is called a geodetic iteration number  $gin(G)$ . They observe an arbitrary set  $S \subset V(G)$ . We introduce here a concept of the convex pre-hull number that is between the two mentioned. For the last case, we observe all convex sets  $C$  of a graph  $G$ . The convex pre-hull number is a natural upper bound for the copoint pre-hull number and can be different. We study in this work the convex pre-hull number of the lexicographic product of graphs.

Let  $G$  be a simple undirected finite graph. The shortest path between two vertices  $u$  and  $v$  of  $G$  is called a  $u, v$ -geodesic. The distance  $d_G(u, v)$  between vertices  $u, v \in V(G)$  is the length of an  $u, v$ -geodesic. The diameter of graph  $G$  is denoted by  $diam(G)$  and is the length of a longest geodesic of  $G$ . Vertices  $u$  and  $v$  for which  $d_G(u, v) = diam(G)$  holds are called the *diametrical vertices*. A (geodesic) interval  $I(u, v)$  between  $u, v \in V(G)$  consists of all vertices that belong to all  $u, v$ -geodesics in  $G$ . Note that if we replace a geodesic in the above definition by an induced, a longest, or any path we obtain *induced*  $J(u, v)$ , *longest* or *detour*  $D(u, v)$ , or *all-path*  $A(u, v)$  intervals, respectively.

In general a *convexity* on a set  $X$  is an algebraic closure system  $\mathcal{C}$  on  $X$  and elements of  $\mathcal{C}$  are called *convex sets*. For general convexities see the book of van de Vel [17]. If we concentrate on graphs we can define the (geodesic) convexity in terms of intervals. Namely, a subset  $C$  of  $V(G)$  is (geodesically) *convex* if  $I(u, v) \subseteq C$  for all  $u, v \in C$ . Again we could define the *monophonic* or *induced path* convexity, the *detour* or *longest path* convexity, or the *all-path* convexity in graph  $G$ ; see more in [1,4,5,7,8,13]. In this work we concentrate solely on (geodesic) convexity and we will omit from now on the term geodesic. Note that if  $C$  induces a complete graph in  $G$  or if  $C = V(G)$ , then  $C$  is convex for any graph  $G$ . We call such sets *trivial* convex sets. Let  $A$  be a subset of  $V(G)$ . The convex hull  $ch(A)$  is the smallest convex set that contains  $A$ . Clearly  $ch(A) = A$  if and only if  $A$  is a convex set. For  $v \in V(G)$  let  $C$  be a maximal convex set with respect to inclusion and with the property that  $v \notin C$ . Then  $C$  is called a *copoint* of  $v$  and  $v$  is an *attaching point* of  $C$ . The set of all attaching points of  $C$  is denoted by  $Att(C)$ .

We now define the *pre-hull operator*  $\ell : \mathcal{P}(V(G)) \rightarrow \mathcal{P}(V(G))$  for a connected graph  $G$  as follows:

$$\ell(A) = \bigcup_{u, v \in A} I(u, v)$$

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for every  $A \in \mathcal{P}(V(G))$ . Clearly  $\ell(A) = A$  if and only if  $A$  is convex set. Thus  $\ell$  is in a sense more interesting for nonconvex sets and we can measure with it “how far” a set  $A$  is from being convex in  $G$ . For this, observe that we can express the convex hull of  $A$  using  $\ell$ :

$$\text{ch}(A) = \bigcup_{n \in \mathbb{N}} \ell^n(A)$$

where  $\ell^n(A)$  is defined inductively:  $\ell^n(A) = \ell(\ell^{n-1}(A))$ . Harary and Nieminen [10] defined the *geodetic iteration number* for set  $A$ ,  $\text{gin}(A)$ , as the smallest integer  $n$  for which  $\text{ch}(A) = \ell^n(A)$ , while the *geodetic iteration number* for graph  $G$ ,  $\text{gin}(G)$ , is the maximum of  $\text{gin}(A)$  taken over all subsets  $A$  of  $V(G)$ .

Let  $v$  be an arbitrary vertex of  $G$  and let  $C$  be any convex set in  $G$ . Then  $\ell^n(C \cup \{v\})$  must be convex for some  $n \in \mathbb{N}_0$  since we deal only with finite graphs and we denote the smallest such number as  $r(v; C)$ . In particular  $\ell^{r(v; C)}(C \cup \{v\}) = \ell^{r(v; C)+1}(C \cup \{v\})$ . Note that  $r(v; C)$  can be 0 if  $v \in C$  or  $v \notin C$  and  $C \cup \{v\}$  is convex already. The *convex pre-hull number* of a convex set  $C$  is then

$$\text{cph}(G; C) = \max\{r(v; C) \mid v \in V(G)\}$$

and the *convex pre-hull number* of a graph  $G$  is

$$\text{cph}(G) = \max\{\text{cph}(v; C)\}$$

where the maximum is taken over all convex sets  $C$  in  $G$ . In addition note that we can use “maximum” in the definition since we are interested only in finite graphs. In the case of infinite graphs one must replace “maximum” with “supremum”.

Polat and Sabidussi have in [15] an additional restriction where they defined a pre-hull number only with copoints  $C$ . We will use the term *copoint pre-hull number* for this. More precisely let  $G$  be a connected graph on at least two vertices and let  $C$  be a copoint in  $G$ . Then the *copoint pre-hull number* of  $C$  is

$$\text{ph}(G; C) = \max\{r(v; C) \mid v \in \text{Att}(C)\}$$

and the *copoint pre-hull number* of a graph  $G$  is

$$\text{ph}(G) = \max\{\text{ph}(v; C)\}$$

where the maximum is taken over all copoints  $C$  in  $G$ .

The obvious bounds are  $0 \leq \text{ph}(G) \leq \text{cph}(G) \leq \text{gin} \leq |V(G)| - 2$ . Indeed, the first inequality is a direct consequence of the definition, the second is due to the fact that copoints are convex sets as well, the third inequality holds since convex sets are also sets and the last must hold if  $C$  is a singleton and at each step for the pre-hull operator we add exactly one vertex. It is easy to see that  $\text{gin}(T) = \text{cph}(T) = 1 > 0 = \text{ph}(T)$  for every tree  $T$  on at least three vertices. This already asserts that the above upper bound is not very accurate. Also the third inequality can be strict since  $1 = \text{cph}(Q_3) < \text{gin}(Q_3) = 2$  for a cube  $Q_3$ . To see the last case, take for  $A$  three vertices of  $Q_3$  that are pairwise at the distance 2.

In the remainder of this section we define some standard terminology. For a graph  $G$  and a vertex  $v \in V(G)$  we use the standard notation

$$N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$$

for the *open neighborhood* of vertex  $v$ ,  $N_G[v] = N_G(v) \cup \{v\}$  is the *closed neighborhood* of  $v$ , and

$$S_i^G(g) = \{u \in V(G) \mid d_G(u, g) = i\}$$

for the  *$i$ th sphere* of  $g$  in  $G$ . We denote as  $\langle C \rangle$  an induced subgraph on  $C \subseteq V(G)$  vertices.

## 2. Lexicographic product

(Standard) graph products are now well studied procedures of graph enlargement. They have been investigated with respect to their structure, the (non)uniqueness of their factors as well as the decomposition algorithms and their complexity. For general results see the book [11] but also more recent papers on that topic [9,12]. Another standard approach to graph products is to find some properties of the product with respect to the properties of their factors. This approach is used also in this work for the lexicographic product. For some small collections of various results of this type see some recent papers [2,3,6,14,16,18].

The *lexicographic product* of graphs  $G$  and  $H$ ,  $G \circ H$  (also  $G[H]$ ), has the vertex set  $V(G) \times V(H)$ . Two vertices  $(g, h)$  and  $(g', h')$  are adjacent in  $G \circ H$  if  $uv \in E(G)$  or  $u = v$  and  $xy \in E(H)$ . For  $v \in V(H)$ ,  $G^v = \{(u, v) \in V(G \circ H) : u \in V(G)\}$  is a  $G$  fiber in  $G \circ H$  and for  $u \in V(G)$ ,  ${}^uH = \{(u, v) \in V(G \circ H) : v \in V(H)\}$  is an  $H$  fiber. Note that the subgraph of  $G \circ H$  induced on  $G^v$  is isomorphic to  $G$  and the subgraph of  $G \circ H$  induced on  ${}^uH$  is isomorphic to  $H$ . Note also that lexicographic product is associative, and has  $K_1$  for a unit, but is not commutative; cf. [11]. A map  $p_G : V(G \circ H) \rightarrow V(G)$  defined by  $p_G((g, h)) = g$  is called the *projection* of  $G * H$  to the first factor  $G$ . Similarly we define the projection  $p_H$  to the second factor  $H$ . Projections  $p_G$  and  $p_H$  can be in a natural way extended from maps on vertices to maps  $p'_G$  and  $p'_H$ , respectively, between graphs  $G \circ H$  and  $G$  and  $H$ , respectively.

Recently in [1], convex sets of lexicographic products have been described. In most cases there are only trivial convex sets in the lexicographic product of arbitrary graphs, but there is an exception if the second factor is complete. To recall this result we need the following concept. A vertex  $u$  of a graph  $G$  is a  $\Delta$ -vertex if  $u$  is adjacent to two nonadjacent vertices. (The notation reflects the fact that  $u$  is the middle vertex of an induced path on three vertices.) Note that all the other vertices of  $G$  have complete (open) neighborhoods. A subset  $Y$  of  $V(G \circ H)$  will be called  $\Delta$ -complete if  ${}^g H \cap Y = {}^g H$  holds for any  $\Delta$ -vertex  $g$  of  $p_G(Y)$ .

**Theorem 2.1.** *Let  $G \circ H$  be a nontrivial, connected lexicographic product. Then  $C$  is a nontrivial convex set of  $V(G \circ H)$  if and only if the following conditions hold:*

- (i)  $p_G(C)$  is convex in  $G$ ,
- (ii)  $C$  is  $\Delta$ -complete, and
- (iii)  $H$  is complete.

In view of Theorem 2.1 we divide the discussion on the copoint and the convex pre-hull numbers of the lexicographic product into two sections, according to whether a second factor is or is it not a complete graph. Before that we state the main result.

We say that  $G$  is a  $\Delta$ -diametrical graph, or  $\Delta$ -D graph for short, if there exists such a pair of diametrical vertices  $g$  and  $\bar{g}$  with  $\text{diam}(G) = d(g, \bar{g})$  in  $G$  that  $g$  is a  $\Delta$ -vertex with two nonadjacent neighbors  $g'$  and  $g''$  for which every vertex of  $N = N_G(g') \cap N_G(g'')$  is a diametrical vertex with  $\bar{g}$ . In particular note that if  $\bar{g} \in N$  for each pair of diametrical vertices  $g$  and  $\bar{g}$  of  $G$ , then  $G$  is not a  $\Delta$ -diametrical graph. Complete graphs and trees are clearly not  $\Delta$ -diametrical graphs.

We can summarize all the result in the following theorem:

**Theorem 2.2.** *Let  $G$  be a connected graph and  $H$  a graph, both with at least two vertices. Then*

$$\begin{aligned} \text{ph}(G \circ H) &= \begin{cases} \text{diam}(G) + 1 & | H \not\cong K_m \text{ and } (G \text{ is a } \Delta\text{-D graph or } G \cong K_m) \\ \text{diam}(G) & | H \not\cong K_m \text{ and } G \not\cong K_m \text{ is not a } \Delta\text{-D graph} \\ \text{ph}(G) & | H \cong K_m; \end{cases} \\ \text{cph}(G \circ H) &= \begin{cases} \text{diam}(G) + 1 & | H \not\cong K_m \text{ and } (G \text{ is a } \Delta\text{-D graph or } G \cong K_m) \\ \text{diam}(G) & | H \not\cong K_m \text{ and } G \not\cong K_m \text{ is not a } \Delta\text{-D graph} \\ \text{cph}(G) & | H \cong K_m. \end{cases} \end{aligned}$$

The proof of this theorem will be given by Theorems 3.1, 4.3 and 4.4 from what follows. Note that there is no difference between  $\text{cph}(G \circ H)$  and  $\text{ph}(G \circ H)$  if  $H$  is noncomplete and it stays the same (with respect to  $G$ ) if  $H$  is complete. Also they have no upper bound with the respect to the convex or copoint pre-hull numbers of the factors if the second factor is not a complete graph. Indeed it is not hard to find an example ( $P_n \circ P_n$  for instance) with fixed convex and copoint pre-hull numbers ( $\text{cph}(P_n) = 1$  and  $\text{ph}(P_n) = 0$ ) and with growing diameter ( $\text{diam}(P_n) = n - 1$ ). Hence the next corollary is clear.

**Corollary 2.3.** *Let  $G$  be a connected graph and  $H$  a noncomplete graph. Then the convex and the copoint pre-hull numbers of  $G \circ H$  are not bounded from above by any function of the convex and the copoint pre-hull numbers of their factors.*

### 3. $H \cong K_m$

Let  $H \cong K_m$ ,  $G$  be a graph on at least two vertices and let  $(g, h) \in V(G \circ K_m)$ . We first concentrate on copoints in  $G \circ K_m$ . Let  $C_g$  be a copoint of  $g$  in  $G$ . We have two possibilities: either  $C_g \cup \{g\}$  is or is it not a convex set of  $G$ . If it is not, it is not hard to see that  $C_g \times V(K_m)$  satisfies all conditions of Theorem 2.1 and that it is a copoint for any vertex  $(g, h)$ ,  $h \in V(K_m)$ . For the first case, note that  $g$  is not a  $\Delta$ -vertex of  $\langle C_g \cup \{g\} \rangle$  if  $C_g \cup \{g\}$  is convex in  $G$ . Moreover, for every convex set  $C$  in  $G$  for which  $C_g \cup \{g\} \subset C$ ,  $g$  is a  $\Delta$ -vertex in  $\langle C \rangle$ . If not, then every neighbor  $g' \in C \setminus C_g$  of  $g$  would induce a complete subgraph with  $N_{\langle C \rangle}(g)$ —a contradiction to the maximality of  $C_g$ . Then for a fixed vertex  $(g, h)$ ,  $h \in V(K_m)$ , the set  $C' = ((C_g \cup \{g\}) \times V(K_m)) \setminus \{(g, h)\}$  is a copoint of  $(g, h)$ . Note that for the last case we have  $\ell(\{(g, h)\} \cup C') = \{(g, h)\} \cup C'$ , which only implies  $\text{ph}(G \circ H) \geq 0$ . Clearly there are no other copoints in  $G \circ K_m$ .

If sets  $C_g \cup \{g\}$  are convex for any copoint  $C_g$  of  $g$  in  $G$ , this means that  $\text{ph}(G \circ K_m) = \text{ph}(G) = 0$ . Thus we can restrict ourselves to a copoint  $C_g$  of  $g$  in  $G$  where  $C_g \cup \{g\}$  is not convex. Then  $C = C_g \times V(K_m)$  is a copoint of  $(g, h)$  for any  $h \in V(K_m)$ . We can choose  $g$  and  $C_g$  such that  $\text{ph}(G)$  is achieved by  $g$  and  $C_g$ . Note that if  $g'$  was covered in the  $i$ th step,  $i > 0$ , of the pre-hull operator in  $G$  for  $g$  and  $C_g$ , the whole  ${}^g K_m$  fiber is covered in the  $i$ th step of the pre-hull operator for  $(g, h)$  and  $C$ . Thus  $\text{ph}(G \circ K_m) \geq \text{ph}(G)$ . On the other hand, we cannot have more, since after  $\text{ph}(G)$  steps we have a convex set in  $G \circ K_m$  and thus also in  $G$  by Theorem 2.1. Hence  $\text{ph}(G \circ K_m) = \text{ph}(G)$  in all cases.

For the same reasons we can see that the convexity pre-hull number of the lexicographic product is the same as the convexity pre-hull number of the first factor. This is true since the projection  $p_G(C)$  is convex in  $G$  by Theorem 2.1 for any convex  $C$  set of  $G \circ K_m$ . We only start with vertex  $g$  and a convex set  $C_g$  for which  $\text{cph}(G)$  is achieved. Thus we have proved:

**Theorem 3.1.** *Let  $G$  be a connected graph with at least two vertices and  $H$  a complete graph. Then*

$$\text{ph}(G \circ H) = \text{ph}(G) \quad \text{and} \quad \text{cph}(G \circ H) = \text{cph}(G).$$

#### 4. $H \not\cong K_m$

Throughout this subsection we have  $H \not\cong K_m$ . For the upper bound we need the following observation: if  ${}^g H \subseteq C$ , then  ${}^{g'} H \subseteq \ell(C)$  for any  $g' \in N_G(g)$ . This follows from the fact that in  $\langle {}^g H \rangle \cong H \not\cong K_m$  there exist two vertices at a distance of at least 2. On the other hand,  $I_{G \circ H}((g, h), (g, h'))$  contains  ${}^{g'} H$  for any  $g' \in N_G(g)$  whenever  $h$  and  $h'$  are nonadjacent vertices of  $H$  and the observation is clear. Similarly,  $I_{G \circ H}((g, h), (g', h'))$  contains  ${}^{g''} H$  for any  $g'' \in I_G(g, g') \setminus \{g, g'\}$ . Hence we have two possibilities for any convex set  $C$  (that induce a complete graph): either  $\ell(\{(g, h)\} \cup C) = \{(g, h)\} \cup C$  or all vertices of some fiber  ${}^{g'} H$  will be in  $\ell(C)$ . We can ignore the first case since we need to find the maximum for the two pre-hull numbers and in the second case, by the observation, the maximum is achieved when  $g'$  is a diametrical vertex. Thus by the observation there can be at most  $\text{diam}(G \circ H) + 1$  steps for the pre-hull operator  $\ell$  and we have proved the following upper bound.

**Theorem 4.1.** *Let  $G$  be a connected graph with at least two vertices and  $H$  a noncomplete graph. Then*

$$\text{diam}(G \circ H) + 1 \geq \text{cph}(G \circ H) \geq \text{ph}(G \circ H).$$

For the lower bound let  $g$  be a diametrical vertex, and choose any maximal complete subgraph  $K^g$  of  $G$  that contains  $g$  and let  $h \in V(H)$  be such a vertex that is not contained in all maximal complete subgraphs of  $H$  (such a vertex exists since  $H \not\cong K_m$ ). Suppose that  $h$  is not contained in a maximal complete subgraph  $K^h$  of  $H$ . Then  $C = V(K^g) \times V(K^h)$  clearly forms a copoint of  $(g, h)$  in  $G \circ H$ . For  $(g, h)$  and its copoint  $C$  we need exactly  $\text{diam}(G)$  steps for the pre-hull operator to stabilize. Namely, after the first step all vertices of the form  $(v, y)$  are added to  $C \cup \{(g, h)\}$  where  $v \in N_G(g)$  and  $y \in V(H)$  (if they are not already in  $C \cup \{(g, h)\}$ ). There is at least one such vertex  $(v, h)$  for any  $v \in N_G(g) \cap V(K^g)$  for instance) and at least one is clearly on a shortest  $(g, h)$ ,  $(g, h')$ -path for  $h' \in V(K^g)$  that is not adjacent to  $h$  in  $H$ . Moreover at step  $i > 1$  we add all vertices  $S_i^{G \circ H}((g, h))$ , and for  $i = 2$  also all vertices of  ${}^g H$  that are not in  $\ell(\{(g, h)\} \cup C)$  are also in  $\ell^2(\{(g, h)\} \cup C)$ . Thus  $\ell^{\text{diam}(G)}(\{(g, h)\} \cup C) = V(G \circ H)$  and we have the lower bound for the copoint pre-hull number.

**Theorem 4.2.** *Let  $G$  be a connected graph with at least two vertices and  $H$  a noncomplete graph. Then*

$$\text{cph}(G \circ H) \geq \text{ph}(G \circ H) \geq \text{diam}(G \circ H).$$

By Theorems 4.1 and 4.2 we see that the difference can only be 1 between the two pre-hull numbers in the lexicographic product when  $H$  is not a complete graph. Next we discuss for which of them the convex and copoint pre-hull number equals  $\text{diam}(G \circ H)$  and for which it equals  $\text{diam}(G \circ H) + 1$ . For this we describe all the maximal complete subgraphs of  $G \circ H$  where  $H \not\cong K_m$ , since by Theorem 2.1 they are the only copoints in  $V(G \circ H)$  (for any vertex outside of them). Let  $K$  be a subset of  $V(G \circ H)$  with the following properties: the projection  $p_G(K)$  induces a maximal complete subgraph of  $G$  and for each vertex  $g \in p_G(K)$  the intersection of  ${}^g H$  and  $K$  induces a maximal complete graph in  ${}^g H$ . Clearly vertices of  $K$  induce a complete subgraph of  $G \circ H$ . Furthermore this complete subgraph is maximal by the maximality of  $p_G(K)$  and that of  ${}^g H \cap K$  for any  $g \in p_G(K)$  and is thus a copoint for any vertex from  $V(G) \setminus K$ . Moreover every copoint of  $V(G \circ H)$  has such a structure; otherwise we would have a contradiction with the maximality again.

**Theorem 4.3.** *Let  $G$  be a  $\Delta$ -diametrical graph and  $H$  a noncomplete graph. Then*

$$\text{cph}(G \circ H) = \text{ph}(G \circ H) = \text{diam}(G \circ H) + 1.$$

**Proof.** Let first  $G$  be a  $\Delta$ -diametrical graph with  $g, g', g''$ , and  $\bar{g}$  appropriate vertices. Let  $g \in K^g \subset V(G)$  where  $\langle K^g \rangle$  is a maximal complete subgraph of  $G$ . Clearly  $g'$  and  $g''$  are not both in  $K^g$ . We may assume without loss of generality that  $g' \in K^g$  and  $g'' \notin K^g$ . Let  $C$  be a copoint of  $(g'', h)$  with property  $p_G(C) = K^g$ . But then

$$\ell(\{(g'', h)\} \cup C) = \{(g'', h)\} \cup C \cup \bigcup_{g_k \in N} {}^{g_k} H$$

and since every vertex from  $N = N_G(g') \cap N_G(g'')$  is a diametrical vertex with  $\bar{g}$  we have for  $i > 1$

$$\ell^{i+1}(\{(g'', h)\} \cup C) = \ell^i(\{(g'', h)\} \cup C) \cup \bigcup_{g_k \in S_i(g)} {}^{g_k} H.$$

In other words  $\ell(\{(g'', h)\} \cup C)$  contains the hole fiber  ${}^a H$  only if  $a$  is the diametrical vertex with  $\bar{g}$  and we still need  $\text{diam}(G \circ H)$  steps for the pre-hull  $\ell$  operator to include  ${}^{\bar{g}} H$ . Thus we have found a copoint  $C$  and its attaching vertex  $(g'', h)$  such that

$$\text{cph}(G \circ H) \geq \text{ph}(G \circ H) \geq \text{diam}(G \circ H) + 1$$

for any  $\Delta$ -diametrical graph  $G$  and noncomplete graph  $H$ . This inequality combined with Theorem 4.1 yields the desired result.  $\square$

**Theorem 4.4.** Let  $G$  be a non- $\Lambda$ -diametrical and noncomplete graph and  $H$  a noncomplete graph. Then

$$\text{cph}(G \circ H) = \text{ph}(G \circ H) = \text{diam}(G \circ H).$$

**Proof.** Let now  $G$  be a non- $\Lambda$ -diametrical graph,  $C$  any convex set of  $V(G \circ H)$ , and  $(g', h)$  a vertex of  $G \circ H$  not contained in  $C$ . In addition let  $G$  be noncomplete. As before, we may assume that  $\ell(\tilde{g}, h)$  and  $C$  are chosen such that  $\ell(\tilde{g}, h) \cup C$  contains at least one fiber  ${}^gH$ . If at least one such  $g$  is not a diametrical vertex then  $V(G \circ H) \subseteq \ell^{\text{diam}(G \circ H)}(\{\tilde{g}, h\} \cup C)$  and we are done. If  $g$  is a diametrical vertex of  $G$  with  $\text{diam}(G) = d(g, \bar{g})$  we have two possibilities:  $g$  is not a  $\Lambda$ -vertex or there exists a vertex  $u \in N = N_G(g') \cap N_G(g'')$  with  $d_G(u, \bar{g}) < \text{diam}(G)$  for some nonadjacent neighbors  $g'$  and  $g''$  of  $g$ .

Suppose first that  $g$  is not a  $\Lambda$ -vertex. Then  $N_G[g]$  induces a complete subgraph of  $G$  and  $g$  is not contained in any interval  $I_G(a, b)$  for  $g \neq a, b$ . Since  ${}^gH \subseteq \ell(\{\tilde{g}, h\} \cup C)$  there must exist a vertex  $(\tilde{g}, h') \in C$  such that  $d_H(h, h') > 1$ . But then  ${}^vH \subseteq \ell(\{\tilde{g}, h\} \cup C)$  for every  $v \in N_G(\tilde{g})$ . Thus we are done if there is a nondiametrical vertex in  $N_G[\tilde{g}]$ . Otherwise for each  $v \in N_G[\tilde{g}]$  there exists  $\bar{v}$  with  $\text{diam}(G) = d(v, \bar{v})$  but also  $v' \in N_G[\tilde{g}]$  with  $\text{diam}(G) = d(v', \bar{v}) + 1$ . Then every vertex  $x$  of  $G$  is in some  $S_{\text{diam}(G)-1}^G(v)$  for  $v \in N_G[\tilde{g}]$  and thus  ${}^xH$  is contained in  $\ell^{\text{diam}(G)}(\{\tilde{g}, h\} \cup C)$ .

For the other case we assume that there exists a vertex  $u \in N$  with  $d_G(u, \bar{g}) < \text{diam}(G)$  for some nonadjacent neighbors  $g'$  and  $g''$  of  $g$  where  $\text{diam}(G) = d(g, \bar{g})$ . We are done again if  $u$  is not a diametrical vertex. Otherwise let  $d_G(u, \bar{u}) = \text{diam}(G)$ . If  $d_G(g, \bar{u}) < \text{diam}(G)$ , we have  ${}^uH \in \ell^{\text{diam}(G)}(\{\tilde{g}, h\} \cup C)$  and we are done. If  $d_G(g, \bar{u}) = \text{diam}(G)$  then  $g'$  and  $g''$  are two nonadjacent neighbors of  $g$  and there exists thus a vertex  $u' \in N$  such that  $d_G(u', \bar{u}) < \text{diam}(G)$ . Since  ${}^{u'}H \in \ell(\{\tilde{g}, h\} \cup C)$ , also  ${}^uH \in \ell^{\text{diam}(G)}(\{\tilde{g}, h\} \cup C)$  and we have  $\text{cph}(K_n \circ H) = \text{ph}(K_n \circ H) = 2$  by Theorem 4.1. Hence the complete graphs are the only non- $\Lambda$ -diametrical graphs for which the upper bound from Theorem 4.1 is achieved.  $\square$

We still have to check both pre-hull numbers if  $G$  is a complete graph. Let  $h$  be a  $\Lambda$ -vertex of  $H$  with respect to nonadjacent neighbors  $h'$  and  $h''$ . Then  $(g, h')$  and  $(g, h'')$  are not both in a copoint  $C$  with respect to  $(g, h)$  for some  $g \in V(G)$ . Say that  $(g, h'')$  is not in  $C$ . But then  $(g, h'') \notin \ell(\{(g, h)\} \cup C)$  and  ${}^gH \subseteq \ell(\{(g, h)\} \cup C)$  for every  $g' \in V(G)$  and  $g' \neq g$ . Thus  $(g, h'') \in \ell^2(\{(g, h)\} \cup C)$  and we have  $\text{cph}(K_n \circ H) = \text{ph}(K_n \circ H) = 2$  by Theorem 4.1. Hence the complete graphs are the only non- $\Lambda$ -diametrical graphs for which the upper bound from Theorem 4.1 is achieved.

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